The role of the equation of state and the space-time dimension in spherical collapse

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(Dated: February 7, 2008)

We study the spherically symmetric collapse of a fluid with non-vanishing radial pressure in higher dimensional space-time. We obtain the general exact solution in the closed form for the equation of state ($P_r = \gamma\rho$) which leads to the explicit construction of the root equation governing the nature (black hole versus naked singularity) of the central singularity. A remarkable feature of the root equation is its invariance for the three cases: $(D+1,\ \gamma=-1)$, $(D,\ \gamma=0)$ and $(D-1,\ \gamma=1)$ where D is the dimension of space-time. That is, for the ultimate end result of the collapse, D-dimensional dust, $D+1$ - AdS (anti de Sitter)-like and $D-1$ - dS-like are absolutely equivalent.

PACS numbers: 04.50.+h, 04.70.Bw, 04.20.Dw, 04.20.Jb

The tussle between black hole and naked singularity as the ultimate end product of gravitational collapse is one of the outstanding problems of classical general relativity (GR). In spite of vigorous activity over two decades, we are far from answering the question in a satisfactory manner. In fact we have no more than a few conjectures, such as Penrose's cosmic censorship conjecture (CCC) [1] (see [2, 3] for reviews on the CCC) and Thorne's hoop conjecture [4], to go by. On the other hand gravitational collapse under fairly general conditions leads to singularity is very well established, thanks to the elegant and powerful singularity theorems of Penrose and Hawking [5].

The next important question is, whether or not singularity so formed will causally influence any regular part of the space-time. The CCC essentially says that a naked singularity (NS) which is formed by evolution of *regular initial data* will be completely shielded from the external view by an event horizon. Such a singularity can be visible only to observers who fall through the event horizon into the black hole (BH). That means light rays can emanate from singularity but are completely blocked by the event horizon and hence they could only lay bare to observers who are co-falling with the collapsing star and never to external observers. This is the weak CCC, while the strong CCC prohibits its visibility by any observer. That means no light rays emanate out of singularity, i.e., it is never naked. In the precise mathematical terms it demands that space-time be globally hyperbolic (for a given initial data, the dynamical evolution is uniquely predictable). Existence of NS would therefore mean failure of global hyperbolicity and thereby deterministic dynamics. This is why CCC is an essential ingredient in a number of key theorems in GR, such as the black hole area and the uniqueness theorems, and the positivity of mass theorem.

Of the two versions, the weak CCC seems to hold ground while there do exist certain counter examples seriously challenging the strong CCC [3]. It has been shown that it is possible to develop NS from regular initial data. The simplest setting for this is the spherical dust collapse described by the Tolman-Bondi metric, which has been extensively studied [6]. We have gained good bit of insight into the formation, visibility and causal structure of the dust collapse singularities. All these works neglect pressure which may play non trivial role in the final outcome of the collapse. From this perspective, gravitational collapse of perfect fluid has been studied [7] to understand the role of pressure and the equation of state. It however turns out that the presence of pressure does not qualitatively alter the final outcome. In particular, the case of radial pressure with vanishing tangential pressure has been analyzed by Gonçalves and Jhingan [8] and it has been shown, for an equation of state $P_r = \gamma\rho$, that the effect of radial pressure for non negative γ is to shrink the parameter window in the initial data space giving rise to naked singularity. However, it could not prevent the formation of naked singularity. On the other hand, when $\gamma = -1$, it is always NS, completely violating CCC.

In recent years, the string theory has provoked explosive interest among theoretical physicists in studying physics in higher dimensions (HD) [9]. While gravitational collapse has been originally studied in four dimensions (4D), there have been several attempts to study it in HD space-time [10, 11, 12, 13, 14, 15]. Interestingly it turns out that as dimension increases the parameter window for naked singularity shrinks continuously. Recently, it has been conjectured that for a marginally bound dust collapse, with initial density profile sufficiently differentiable or smooth ($\rho_1 = 0$), the CCC is always respected in HD with $D \geq 6$ [15]. This is however not true for the profile with ρ_1 non vanishing, where the increase in D only results in shrinking of the parameter window lead-

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ing to NS and in the non-marginally bound case even the condition $\rho_1 = 0$ does not save CCC.

In this study we would like to examine the role of radial pressure and extra dimensions for the spherical collapse. Of particular interest would be the relative strength of the contributions due to the increase in dimension which favors BH and the negative γ which favors NS. For this, following the method of Ref. [8], we first obtain an exact solution of the Einstein equation in HD for a fluid with radial pressure, satisfying the equation of state, $P_r = \gamma\rho$, and vanishing tangential pressure. We shall then analyze the tug of war between BH and NS. The most remarkable result that emerges from this analysis is the interplay between the dimension D of space-time and the equation of state parameter γ. It turns out that the final outcome of the collapse for dust in D, $\gamma = 1$ in $D-1$ and $\gamma = -1$ in $D+1$ dimensions is the same. That is these three cases are indistinguishable under gravitational collapse. This is the main result that we would like to share through this communication.

We write, for the $D = n+2$ dimensional spherically symmetric space-times, the metric in the comoving coordinates

$$ds^2 = e^{2\psi}dt^2 - e^{2\vartheta}dr^2 - Y^2 d\Omega^2, \tag{1}$$

where ψ, ϑ and Y are function of r and t, and $d\Omega^2$ is the metric on an n-sphere.

The Einstein equation, $G_{ab} = -\kappa T_{ab}$ where $T^a_b = \mathrm{diag}(\rho, -P_r, -P_\theta, . . . \; - P_\theta)$, after some manipulations lead to the following system of equations:

$$\begin{aligned} m' &= \kappa\frac{(n-1)}{n}Y^n Y'\rho, && (2)\\ \dot{m} &= -\kappa\frac{(n-1)}{n}Y^n \dot{Y} P_r, && (3)\\ \psi'(\rho + P_r) &= \left[n(P_r - P_\theta)\frac{Y'}{Y} + {P_r}'\right] && (4) \end{aligned}$$

where

$$\begin{aligned} k(t,r) &= 1 - e^{-2\vartheta}Y'^2, && (5)\\ m(t,r) &= \frac{n-1}{2}Y^{n-1}\left[e^{-2\psi}\dot{Y}^2 + k(t,r)\right]. && (6) \end{aligned}$$

Here dot and prime stand respectively for the differentiation with respect to t and r. The 'conservation' equation $\nabla_a T^a_b = 0$ reads

$$\begin{aligned} \dot{\vartheta} &= \frac{\dot{Y}'}{Y'} - \psi'\frac{\dot{Y}}{Y'} && (7)\\ \dot{\rho} &= -(\rho + P_r)\left(\dot{\vartheta} + n\frac{\dot{Y}}{Y}\right) + n(P_r - P_\theta)\frac{\dot{Y}}{Y}, && (8) \end{aligned}$$

In this model we take zero tangential pressure and the equation of state for the radial pressure, $P_r = \gamma\rho$ with $-1 < \gamma < 1$. Setting the tangential pressure $P_\theta = 0$ in Eq. (7), yields

$$\psi'(\rho + P_r) = \left[nP_r\frac{Y'}{Y} + {P_r}'\right]. \tag{9}$$

Further we consider the marginally bound case which means $k(t,r) = 0$ and so we get $\dot{\vartheta} = \dot{Y}'/Y'$. Then the above equation implies $\psi = \psi(t)$ which could be absorbed by redefining t to proper time τ via $\tau = \int e^{\psi(t)}dt + f(r)$. We thus obtain the general solution

$$Y(\tau, r) = A^{\frac{1}{(n+\gamma+1)}} r^{(n+\gamma-1)/(n+\gamma+1)} \left[\tau_0(r) - \tau\right]^{\frac{2}{(n+\gamma+1)}}, \tag{10}$$

with

$$\rho = C\frac{r^{n-2}}{Y^n}\left(\frac{r}{Y}\right)^{1+\gamma} \tag{11}$$

where $\tau_0(r) = r/\sqrt{A}$,

$$A = \frac{(n+\gamma+1)^2}{4}\left(\frac{2\kappa}{n}C\right)$$

and C is a constant. Here, we note that the solution (10), as in the 4D case, is exact only for $\gamma = -1,\ 0$ and approximate otherwise. For other value of γ, $G_{\theta_1\theta_1} \neq 0$ and it in fact reads as

$$P_\theta \propto (1+\gamma)\gamma r^{2(n+\gamma-1)/(n+\gamma+1)}\mathcal{F}(t,r), \tag{12}$$

where $\mathcal{F}(t,0) \propto t^{-(n+\gamma+3)/(n+\gamma+1)}$. It vanishes as r goes to zero. For $\gamma = 0, -1$, we have the exact solution else it is approximately valid close to the central singularity. This is precisely the region of interest as singularity is approached. The weak energy condition which requires $\rho \geq 0$, $(\rho + P_r) \geq 0$, $(\rho + P_\theta) \geq 0$ is clearly satisfied.

The apparent horizon is formed when the boundary of trapped surface is formed at $(n-1)Y^{(n-1)} = 2m$. The corresponding time $\tau = \tau_{\mathrm{ah}}$ is given by

$$\tau_{\mathrm{ah}}(r) = \tau_0(r)\left[1 - \Theta^{(n+\gamma+1)/(n+\gamma-1)}\right], \tag{13}$$

with $\Theta = \sqrt{2\kappa C/n}$. As in the 4D case, it can be shown that $\tau > \tau_{\mathrm{ah}}$ for all $r > 0$ and $\tau_0(0) = \tau_{\mathrm{ah}}(0)$ at $r = 0$. It then follows that only the central singularity at $r = 0$ could be naked while the others with $r > 0$ are all censored. For studying the causal structure of the singularity, we follow the outgoing radial null geodesics and check whether some of them meet the singularity in the finite past. The equation of radial null geodesics is

$$\frac{d\tau}{dr} = \pm Y' = \pm\frac{Y}{r}\frac{1}{(n+\gamma+1)}\left[n + \gamma - 1 + 2\left(1 - \frac{\tau}{\tau_0}\right)^{-1}\right]. \tag{14}$$

Along the outgoing radial null geodesics we have

$$\frac{dY}{dr} = Y' + \dot{Y}\left(\frac{d\tau}{dr}\right) = Y'(1 + \dot{Y}). \tag{15}$$

Using the standard procedure, we introduce the auxiliary variables u, X :

$$\begin{aligned} u &= r^\alpha,\ \alpha > 0, && (16)\\ X &= \frac{Y}{u}. && (17) \end{aligned}$$

In the limit of approach to the singularity we write

$$X_0 = \lim_{Y\to 0, u\to 0} \frac{Y}{u} = \lim_{Y\to 0, u\to 0} \frac{dY}{du} = \lim_{Y\to 0, r\to 0} \frac{1}{\alpha r^{\alpha-1}} \frac{dY}{dr}$$
$$= \lim_{Y\to 0, r\to 0} \frac{1}{\alpha r^{\alpha-1}} Y'(1+\dot{Y}). \quad (18)$$

In order to obtain the root equation, we first obtain an explicit expression for Y'. Now, from Eq. (10), we have

$$\tau(Y,r) = \frac{r}{\sqrt{A}}\left[1 - \left(\frac{Y}{r}\right)^{(n+\gamma+1)/2}\right]. \quad (19)$$

By differentiating Eq. (19) with respect to r, we obtain:

$$Y'(Y,r) = \frac{Y}{r}\frac{1}{(n+\gamma+1)}\left[n+\gamma-1+2\left(\frac{r}{Y}\right)^{(n+\gamma+1)/2}\right]. \quad (20)$$

We insert Eq. (20) into (18) to get

$$X_0 = \lim_{r\to 0} \frac{X}{\alpha(n+\gamma+1)}\Big[\left[n+\gamma-1+2\frac{r^{(1-\alpha)\times(n+\gamma+1)/2}}{X^{(n+\gamma+1)/2}}\right]$$
$$\times\left[1-\Theta\frac{r^{(1-\alpha)(n+\gamma-1)/2}}{X^{(n+\gamma-1)/2}}\right]\Big]. \quad (21)$$

A self consistent solution occurs for $\alpha = 1$. Therefore the desired root equation becomes

$$y_0^{2(n+\gamma)} + \frac{n+\gamma-1}{2}\Theta y_0^{(n+\gamma+1)} - y_0^{(n+\gamma-1)} + \Theta = 0 \quad (22)$$

where $y_0 = \sqrt{X_0}$. Clearly this equation remains unaltered so long as $n+\gamma$ remains fixed. Since n can take only integral value, $n+\gamma$ could remain fixed only if γ takes integral value, which could only be $0, \pm 1$. The equation remains invariant for $(D-1,\ \gamma = 1)$, $(D,\ \gamma = 0)$ and $(D+1,\ \gamma = -1)$ where $n = D-2$.

The existence of a real positive root to this algebraic equation is necessary and sufficient condition for the existence of NS. The values of the roots give the tangents of the escaping geodesics near the singularity. Thus, the occurrence of positive roots would imply the violation of the strong CCC, though not necessarily of the weak form. Hence in the absence of positive real roots, the collapse will always lead to a black hole. The critical slope would be given by the double root, marking the threshold between BH and NS. It is interesting to see that for each D, there exists a Θ^D_{crit} such that singularities are always naked for all $\Theta \in (0, \Theta^D_{\text{crit}}]$, i.e., for each D there exists a non zero measure set of Θ values giving rise to NS and consequently violating CCC.

From the above root equation, which is the master equation governing the end result, BH v/s NS, follows the main result of the paper that dust in D, de Sitter like in $D-1$ and anti de Sitter like in $D+1$ dimensions are absolutely equivalent for the end result of the collapse. (Here, by de Sitter and anti-de Sitter, we simply mean $\gamma = \pm 1$.) This is a remarkable new result which interestingly hooks space-time dimension with the equation of state parameter. Clearly the above root equation always has a positive root for the 4-dimensional AdS-like ($\gamma = -1$) collapse and hence singularity is in this case always naked. It is however known that increase in D makes gravity stronger and thereby it favors BH against NS indicated by the shrinkage of the NS producing parameter window in the initial data set. Positive pressure also has similar contribution while negative pressure has the opposite effect. That is why for $\gamma = -1$, as D increases both BH/NS could occur while it is all NS for $D = 4$. That is strengthening of gravity is stronger due to increase in D than the opposite negative pressure contribution. The parameter window in the initial data set leading to NS shrinks. Though here we have essentially considered radial pressure, the result would be valid in general for any gravitational collapse with pressure.

SGG and DWD would like to thank IUCAA, Pune for hospitality while this work was done. SGG also thanks to Gridhar Vishwanathan for providing him support in computation.

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